On the Minimum Spanning Tree for Directed Graphs with Potential Weights

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Abstract

In general the problem of finding a miminum spanning tree for a weighted directed graph is difficult but solvable. There are a lot of differences between problems for directed and undirected graphs, therefore the algorithms for undirected graphs cannot usually be applied to the directed case. In this paper we examine the kind of weights such that the problems are equivalent and a minimum spanning tree of a directed graph may be found by a simple algorithm for an undirected graph.

1 Introduction

The problem of finding a minimum spanning tree is well known to graph theorists as well as to programmers who deal with graph theory applications. For an undirected case there are a lot of simple algorithms such as Prim's algorithm [5]. For directed graphs, a general solution also exists, see papers [1, 2, 3] for more information. Furthermore, there are some optimizations for directed and undirected cases based on Fibonacci heaps [4].

In certain problems in physics we deal with directed graphs whose weights by definition satisfy the given relation. Sometimes the properties of weights provide a possibility of simplifying the problem to an undirected case. For example, consider a directed graph whose weights satisfy the equation

$$Q_{ij} = \varphi_{ij} - \varphi_{ii}. \tag{1}$$

The main result of this paper is that for the graph whose a weight matrix is Q with positive entries a minimum spanning tree may be found by a simple algorithm for a corresponding undirected graph whose a weight matrix is symmetric matrix φ.

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One can think about the weight φ_{ij} as a height of a potential barrier which has to be surmounted in order to get to point i from point j. From this point of view φ_{ii} is a local minimum of a potential well. So weights which satisfy Equation (1) will be called *potential*.

2 Definitions

Depending on the problem, directed trees may be defined in two ways. Usually it is supposed that in-degree $id(v) \le 1$ for all vertices v. However we define tree such that out-degree $od(v) \le 1$ and $od(v) = 1$ iff v is the root. We will denote $G(V, E, \omega)$ for an undirected graph where V is a vertex set, E is an edge set and ω is a weight matrix corresponding to the edge set. Clearly ω is always symmetric. In the case of a directed graph we will use the same notation except a prime to denote a directed graph and we will use A instead of E in order to note that in the directed case we will have an arc set. Also in the undirected case the weight matrix is not neccessary symmetric.

3 Minimum Spanning Trees

Suppose we have an undirected graph $G(V, E, \varphi)$, where φ is a symmetric matrix whose diagonal entries are not neccessary equal to zero, but inequations $\varphi_{ii} < \varphi_{ik}$ and $\varphi_{ii} < \varphi_{ki}$ hold for all $k \ne i$. For every G we can define a directed graph $G'(V, A, Q)$ where Q is a weight matrix which satisfies Equation (1). We must find a tree T which minimizes the following expression

$$w(T) = \sum_{(i,j)\in T} Q_{ij}. \tag{2}$$

This tree is called a minimum spanning tree for the weighted directed graph G'. The minimum spanning tree for undirected graph G is defined in a similar way.

For this class of weighted directed graphs the following proposition can be posed.

Proposition 1. *A minimum spanning tree of a directed graph G' coincides with a minimum spanning tree for an undirected graph G with a root in vertex k for which φ_{kk} is minimum.*

Proof. The minumum for (2) can be written, according for Q weights properties in following form:

$$\min_{T \subset G'} w(T) = \min_{T \subset G'} \sum_{(i,j)\in T} (\varphi_{ij} - \varphi_{ii}).$$

Allowing for the facts that the number of arcs in spanning tree T equals $|V| - 1$ and the off-diagonal entries do not depend on diagonal ones, the minimum of

the previous expression equals

$$\min_{T\subset G'} \sum_{(i,j)\in T} \varphi_{ij} - \max_k \sum_{i\neq k} \varphi_{ii}.$$

It is clear that if $\varphi_{ii} = 0$ for all i then G' changes[1] to G. Therefore the first summand in the previous expression equals to the weight of the corresponding a minumum spanning tree of the undirected graph G.

$$\min_{T\subset G} \sum_{\{i,j\}\in T} \varphi_{ij} - \max_k \sum_{i\neq k} \varphi_{ii}$$

It follows

$$\min_{T\subset G} \sum_{\{i,j\}\in T} \varphi_{ij} - \sum_{i=1}^{|V|} \varphi_{ii} + \min_k \varphi_{kk}$$

As a result, the minimum spanning tree of G' can be given from the minimum spanning tree of G by fixing a root in vertex k with a minimum value of φ_{kk}. □

4 Conclusion and Future Work

In the previous section it is shown that algorithms for undirected graphs can be applied to a directed case if weights of a directed graph have certain special properties. In the future we plan to get a solution for a more difficult problem, finding the minimum spanning forest for graphs whose weights are potential.

References

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[1] In this case we can change the ordered pair (i, j) to a 2-elements subset $\{i, j\}$.